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Poincaré-type inequalities with $L^p(\log L)^\alpha$-norms for Green's operator

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ABSTRACT

In this paper, we first establish local Poincaré-type inequalities with $L^p(\log L)^\alpha$-norms for Green's operator applied to the solutions of the nonhomogeneous A-harmonic equation. Then, we extend the local results to global versions in the $L^\varphi(\mu)$- domain.

1. Introduction and notation

The purpose of this paper is to extend the Poincaré inequalities for Green's operator applied to the solutions of the nonhomogeneous A-harmonic equation to the case of $L^p(\log L)^\alpha$-norms. Differential forms have become basic tools and have found many applications in different areas of mathematics and physics. For instance, it is well known that an integral $\int_E f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$ is determined by the domain E and the integrand f defined on E. The domain E may be a subset of $\mathbb{R}^k$ with piecewise smooth boundary or a differentiable k-manifold. If we let $\omega = f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$, then ω is a k-form on E. Moreover, if f is differentiable, we call ω a differential k-form on E. Thus, the integral $\int_E f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$ can be denoted by a simple form $\int_E \omega$, which corresponds to line integrals, surface integrals, volume integrals etc., if $k = 1, 2, 3, \ldots$, respectively. The famous Stokes theorem gives the relationship between the integration and the exterior derivative, see [1]. In fact, differential forms can be regarded as the extensions of functions in $\mathbb{R}^k$. So far, many important results for differential forms have been established. However, the study on operators applied to differential forms has just begun, see [2–6].

We first introduce some notation and definitions. Throughout this paper, we always use E to denote an open subset of $\mathbb{R}^n$ and use M to denote a bounded, convex subset of $\mathbb{R}^n$, $n \geq 2$. Let $\wedge^k = \wedge^k(\mathbb{R}^n)$ be the set of all k-forms in $\mathbb{R}^n$ and $D'(E, \wedge^k)$ be the space of all differential k-forms on E. We use $\mathrm{d} : D'(E, \wedge^k) \to D'(E, \wedge^{k+1})$ to denote the differential operator and use $\mathrm{d}^\star : D'(E, \wedge^{k+1}) \to D'(E, \wedge^k)$ to denote the Hodge codifferential operator defined by $\mathrm{d}^\star = (-1)^{nk+1} \star \mathrm{d} \star$ on $D'(E, \wedge^{k+1})$, $k = 0, 1, \ldots, n-1$. Here $\star$ is the Hodge star operator.

Let $\wedge^k M$ be the kth exterior power of the cotangent bundle and $C^\infty(\wedge^k M)$ be the space of smooth k-forms on M. We use $\mathcal{W}(\wedge^k M)$ to denote the space of all $\omega \in L^1_{loc}(\wedge^k M)$ which has generalized gradient and use $\mathcal{H}(\wedge^k M)$ to denote the subspace of $\mathcal{W}(\wedge^k M)$, where the element ω satisfies $\mathrm{d}\omega = \mathrm{d}^\star\omega = 0$, $\omega \in L^p$ for some $1 < p < \infty$.

We consider Green's operator $G : C^\infty(\wedge^k M) \to \mathcal{H}^\perp \cap C^\infty(\wedge^k M)$ defined by assigning $G(\omega)$ as the unique element of $\mathcal{H}^\perp \cap C^\infty(\wedge^k M)$ satisfying Poisson's equation $\Delta G(\omega) = \omega - H(\omega)$, where $H : C^\infty(\wedge^k M) \to \mathcal{H}$ is the harmonic projection operator such that $H(\omega)$ is the harmonic part of ω.

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We call an increasing convex function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$ a Young function. The Orlicz space $L^{\Phi}(E)$ is a set of all measurable functions f on E such that $\int_E \Phi(|f|/\lambda)\mathrm{d}x < \infty$ for some $\lambda = \lambda(f) > 0$ with the nonlinear Luxemburg functional

$$\|f\|_{\Phi} = \inf\left\{\lambda > 0 : \frac{1}{|E|}\int_E \Phi\left(\frac{|f|}{\lambda}\right)\mathrm{d}x \le 1\right\}.$$

Moreover, if Φ is a restrictively increasing Young function, then $L^{\Phi}(E)$ is a Banach space and the corresponding norm $\|\cdot\|_{\Phi}$ is called the Luxemburg norm or Orlicz norm. Setting $\Phi(t) = t^p \log^{\alpha}(\mathrm{e} + t/c)$, we use $L^p(\log L)^{\alpha}(E)$ to denote the Orlicz space $L^{\Phi}(E)$ with norm $\|\cdot\|_{L^p(\log L)^{\alpha}(E)}$, where $1 \le p < \infty$, $\alpha \ge 0$ and $c > 0$ are constants.

In [5], Ding and Liu prove that the norm $\|f\|_{L^p(\log L)^{\alpha}(E)}$ is equivalent to the norm $[f]_{L^p(\log L)^{\alpha}(E)} = (\int_E |f|^p \log^{\alpha}(\mathrm{e} + \frac{|f|}{\|f\|_{p,E}})\mathrm{d}x)^{1/p}$ for any $p \in (0, \infty)$. In this paper, we simply write

$$\|f\|_{L^p(\log L)^{\alpha}(E)} = \left(\int_E |f|^p \log^{\alpha}\left(\mathrm{e} + \frac{|f|}{\|f\|_{p,E}}\right)\mathrm{d}x\right)^{\frac{1}{p}}.$$

Similarly, we use $L^p(E, \wedge^k)$ to denote the space of differential k-forms with coefficients in $L^p(E)$, $1 \le p < \infty$. The norms are given by $\|\omega\|_{p,E} = (\int_E |\omega(x)|^p \mathrm{d}x)^{1/p}$. We write $L^p(\log L)^{\alpha}(E, \wedge^k)$ for the space of all differential k-forms ω on E with

$$\|\omega\|_{L^p(\log L)^{\alpha}(E)} = \left(\int_E |\omega|^p \log^{\alpha}\left(\mathrm{e} + \frac{|\omega|}{\|\omega\|_{p,E}}\right)\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Here $\omega(x) = \sum_I \omega_I(x)\mathrm{d}x_I$ with $\omega_I \in L^p(E)$ for all ordered k-tuples I and $|\omega| = (\sum_I |\omega_I(x)|^2)^{1/2}$.

Consider the nonhomogeneous A-harmonic equation for differential forms

$$\mathrm{d}^{\star}A(x, \mathrm{d}\omega) = B(x, \mathrm{d}\omega), \tag{1.1}$$

where $A : E \times \wedge^k(\mathbb{R}^n) \to \wedge^k(\mathbb{R}^n)$ and $B : E \times \wedge^k(\mathbb{R}^n) \to \wedge^{k-1}(\mathbb{R}^n)$ are two operators satisfying the conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1}, \qquad A(x, \xi)\cdot\xi \ge |\xi|^p \quad \text{and} \quad |B(x, \xi)| \le b|\xi|^{p-1}$$

for almost every $x \in E$ and all $\xi \in \wedge^k(\mathbb{R}^n)$. Here $a, b > 0$ are some constants and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{loc}(E, \wedge^{k-1})$ such that

$$\int_E A(x, \mathrm{d}\omega)\cdot \mathrm{d}\varphi + B(x, \mathrm{d}\omega)\cdot\varphi = 0 \tag{1.2}$$

for all $\varphi \in W^{1,p}_{loc}(E, \wedge^{k-1})$ with compact support. From [2], if ω is a differential form defined in M, then there is a decomposition

$$\omega = \mathrm{d}(T\omega) + T(\mathrm{d}\omega), \tag{1.3}$$

where T is called a homotopy operator. Furthermore, we can define the k-form $\omega_M \in D'(M, \wedge^k)$ by

$$\omega_M = |M|^{-1}\int_M \omega(y)\mathrm{d}y, \quad k = 0, \quad \text{and} \quad \omega_M = \mathrm{d}(T\omega), \quad k = 1, 2, \ldots, n \tag{1.4}$$

for all $\omega \in L^p(M, \wedge^k)$, $1 \le p < \infty$.

2. Local Poincaré inequalities for Green's operator

In this section, we prove the local Poincaré estimates for Green's operator, which need the following results.

We will need the following generalized Hölder inequality.

Lemma 2.1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta}$. If f and g are two measurable functions on $\mathbf{R}^n$, then*

$$\|fg\|_{s,E} \le \|f\|_{\alpha,E}\cdot\|g\|_{\beta,E} \tag{2.1}$$

for any $E \subset \mathbf{R}^n$.

The weak reverse Hölder inequality and its weighted version with respect to L^p-norms were first introduced in [7,8]. Then, Agarwal and Ding developed the inequalities to the case of $L^p(\log L)^{\alpha}$-norms in [9]. The conclusion is stated as following Lemma 2.2.

Lemma 2.2. *Let ω be a solution of the nonhomogeneous A-harmonic equation in E, $\sigma > 1$ and $0 < s, t < \infty$. Then, there exists a constant C, independent of ω, such that*

$$\|\mathrm{d}\omega\|_{L^s(\log L)^\alpha(B)} \le C|B|^{(t-s)/st}\|\mathrm{d}\omega\|_{L^t(\log L)^\beta(\sigma B)} \tag{2.2}$$

for all balls B with $\sigma B \subset E$ and diam$(B) \ge d_0 > 0$. Here d_0 is a fixed constant, $\alpha > 0$ and $\beta > 0$ are any constants.

In [6], Ding proved the following Poincaré inequality for Green's operator applying to differential forms in terms of L^p-norms.

Lemma 2.3. *Let $\mathrm{d}\omega \in L^p(M, \wedge^k)$ be a smooth form and G be Green's operator, $k = 1, 2, \ldots, n$, and $1 < p < \infty$. Then, there exists a constant C, independent of ω, such that*

$$\|G(\omega) - (G(\omega))_B\|_{p,B} \le C|B|\mathit{diam}(B)\|\mathrm{d}\omega\|_{p,B} \tag{2.3}$$

for all balls $B \subset M$.

Theorem 2.4. *Let $\omega \in D'(M, \wedge^k)$ be a solution of the nonhomogeneous A-harmonic equation in a domain $M \subset \mathbf{R}^n$ and $\mathrm{d}\omega \in L^p(M, \wedge^{k+1})$, $k = 0, 1, \ldots, n-1$, $1 < p < \infty$. G be Green's operator. Then, there is a constant C, independent of ω, such that*

$$\|G(\omega) - (G(\omega))_B\|_{L^p(\log L)^\alpha(B)} \le C|B|\mathit{diam}(B)\|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(\sigma B)} \tag{2.4}$$

for all balls B with $\sigma B \subset M$ and diam$(B) \ge d_0$. Here $\alpha > 0$ is any constant and $\sigma > 1$ and $d_0 > 0$ are some constants.

Proof. Let $B \subset M$ be a ball with diam$(B) \ge d_0 > 0$. Choose $\varepsilon > 0$ small enough and a constant C_1 such that

$$|B|^{-\varepsilon/p^2} \le C_1. \tag{2.5}$$

If $\frac{|G(\omega)-(G(\omega))_B|}{\|G(\omega)-(G(\omega))_B\|_{p,B}} \ge 1$, then for above $\varepsilon > 0$, there exists $C_2 > 0$ such that

$$\log^\alpha\left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}}\right) \le C_2\left(\frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p+\varepsilon,B}}\right)^\varepsilon. \tag{2.6}$$

Setting $B_1 = \{x \in B : \frac{|G(\omega)-(G(\omega))_B|}{\|G(\omega)-(G(\omega))_B\|_{p,B}} \ge 1\}$, $B_2 = B \setminus B_1$. By the elementary inequality $|a+b|^s \le 2^s(|a|^s + |b|^s)$, where $s > 0$ is any constant, we have

$$\begin{aligned}
\|G(\omega) - (G(\omega))_B\|_{L^p(\log L)^\alpha(B)} &= \left(\int_B |G(\omega) - (G(\omega))_B|^p \log^\alpha\left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}}\right)\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le 2^{\frac{1}{p}}\left(\int_{B_1} |G(\omega) - (G(\omega))_B|^p \log^\alpha\left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}}\right)\mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad + 2^{\frac{1}{p}}\left(\int_{B_2} |G(\omega) - (G(\omega))_B|^p \log^\alpha\left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}}\right)\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.7}$$

Since

$$\log^\alpha\left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}}\right) \le \log^\alpha(\mathrm{e}+1) \le N, \quad x \in B_2,$$

by Lemma 2.3, we can estimate the second term of (2.7).

$$\begin{aligned}
\left(\int_{B_2} |G(\omega) - (G(\omega))_B|^p \log^\alpha\left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}}\right)\mathrm{d}x\right)^{\frac{1}{p}} &\le N_1\left(\int_{B_2} |G(\omega) - (G(\omega))_B|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\le N_2\|G(\omega) - (G(\omega))_B\|_{p,B} \\
&\le N_3|B|\mathrm{diam}(B)\|\mathrm{d}\omega\|_{p,B} \\
&\le N_4|B|\mathrm{diam}(B)\|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(B)}.
\end{aligned} \tag{2.8}$$

Therefore, we may assume that $\frac{|G(\omega)-(G(\omega))_B|}{\|G(\omega)-(G(\omega))_B\|_{p,B}} \ge 1$ on B. By (2.6) and Lemma 2.3, we have

$$\|G(\omega) - (G(\omega))_B\|_{L^p(\log L)^\alpha(B)} = \left(\int_B |G(\omega) - (G(\omega))_B|^p \log^\alpha\left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}}\right)\mathrm{d}x\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\le C_3\left(\int_B |G(\omega)-(G(\omega))_B|^p\left(\frac{|G(\omega)-(G(\omega))_B|}{\|G(\omega)-(G(\omega))_B\|_{p+\varepsilon,B}}\right)^{\varepsilon}\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le \frac{C_4}{\|G(\omega)-(G(\omega))_B\|_{p+\varepsilon,B}^{\frac{\varepsilon}{p}}}\left(\left(\int_B |G(\omega)-(G(\omega))_B|^{p+\varepsilon}\mathrm{d}x\right)^{\frac{1}{p+\varepsilon}}\right)^{\frac{p+\varepsilon}{p}}\\
&= C_4\|G(\omega)-(G(\omega))_B\|_{p+\varepsilon,B}\\
&\le C_5|B|\mathrm{diam}(B)\|\mathrm{d}\omega\|_{p+\varepsilon,B}. \qquad (2.9)
\end{aligned}
$$

For any $\alpha > 0$, we have

$$
\log^{\alpha}\left(\mathrm{e}+\frac{|\mathrm{d}\omega|}{\|\mathrm{d}\omega\|_{p+\varepsilon,B}}\right) > 1. \qquad (2.10)
$$

Combining (2.9), (2.10) and Lemma 2.2, we obtain

$$
\begin{aligned}
\|G(\omega)-(G(\omega))_B\|_{L^p(\log L)^{\alpha}(B)} &\le C_5|B|\mathrm{diam}(B)\|\mathrm{d}\omega\|_{p+\varepsilon,B}\\
&\le C_6|B|\mathrm{diam}(B)\|\mathrm{d}\omega\|_{L^{p+\varepsilon}(\log L)^{\alpha}(B)}\\
&\le C_7|B|\mathrm{diam}(B)|B|^{\frac{1}{p+\varepsilon}-\frac{1}{p}}\|\mathrm{d}\omega\|_{L^p(\log L)^{\alpha}(\sigma B)}\\
&\le C_8|B|\mathrm{diam}(B)|B|^{-\frac{\varepsilon}{p^2}}\|\mathrm{d}\omega\|_{L^p(\log L)^{\alpha}(\sigma B)}\\
&\le C_9|B|\mathrm{diam}(B)\|\mathrm{d}\omega\|_{L^p(\log L)^{\alpha}(\sigma B)}.
\end{aligned}
$$

Here $\sigma > 1$ is some constant. The proof of Theorem 2.4 has been completed. □

3. Global Poincaré inequalities for Green's operator

In this section, we will give the main result of this paper, that is the weighted global Poincaré inequality for Green's operator with $L^p(\log L)^{\alpha}$-norms.

We call w a weight if w is locally integrable in $\mathbb{R}^n$ and $w > 0$ a.e. We use $L^p(E, \wedge^k, w^{\alpha})$ to denote the weighted L^p space with norm $\|f\|_{p,E,w^{\alpha}} = (\int_E |f|^p w^{\alpha}(x)\mathrm{d}x)^{\frac{1}{p}}$, where $w(x)$ is a weight and α is a real number. Similarly, we say $f \in L^p(\log L)^{\alpha}(E, \wedge^k, w^{\alpha})$, if

$$
\|f\|_{L^p(\log L)^{\alpha}(E,w^{\alpha})} = \left(\int_E |f|^p \log^{\alpha}\left(\mathrm{e}+\frac{|f|}{\|f\|_{p,E}}\right) w^{\alpha}\mathrm{d}x\right)^{\frac{1}{p}} < \infty,
$$

where $w(x)$ is a weight and α is a real number.

The following $A_{r,\lambda}(E)$-weights and $A_r(E)$-weights appear in [10] and [11], respectively. We first give the relationship between the two weights.

Definition 3.1. We say a pair of weights $(w_1(x), w_2(x))$ satisfies the $A_{r,\lambda}(E)$-condition in a set $E \subset \mathbb{R}^n$, write $(w_1(x), w_2(x)) \in A_{r,\lambda}(E)$ for some $\lambda \ge 1$ and $1 < r < \infty$ with $\frac{1}{r}+\frac{1}{r'} = 1$, if

$$
\sup_{B\subset E}\left(\frac{1}{|B|}\int_B w_1^{\lambda}\mathrm{d}x\right)^{\frac{1}{\lambda r}}\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\frac{\lambda r'}{r}}\mathrm{d}x\right)^{\frac{1}{\lambda r'}} < \infty. \qquad (3.1)
$$

Definition 3.2. We say a pair of weights $(w_1(x), w_2(x))$ satisfies the $A_r(E)$-condition in a set $E \subset \mathbb{R}^n$, write $(w_1(x), w_2(x)) \in A_r(E)$ for some $1 < r < \infty$, if

$$
\sup_{B\subset E}\left(\frac{1}{|B|}\int_B w_1\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\frac{1}{r-1}}\mathrm{d}x\right)^{r-1} < \infty. \qquad (3.2)
$$

Proposition 3.3. *If $(w_1(x), w_2(x)) \in A_{r,\lambda}(E)$ for some $\lambda \ge 1$ and $1 < r < \infty$ with $\frac{1}{r}+\frac{1}{r'} = 1$, then $(w_1(x), w_2(x)) \in A_r(E)$.*

Proof. Choose $\frac{1}{r}+\frac{1}{r'} = 1$. From the definition of the A_r-weights and the Hölder inequality, we have

$$
\begin{aligned}
\left(\frac{1}{|B|}\int_B w_1 \mathrm{d}x\right)\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\frac{1}{r-1}}\mathrm{d}x\right)^{r-1} &= \left(\frac{1}{|B|}\right)^r \|w_1\|_{1,B}\,\|1/w_2\|_{\frac{r'}{r},B} \\
&\le |B|^{-r}|B|^{\frac{\lambda-1}{\lambda}}|B|^{\frac{r(\lambda-1)}{\lambda r'}}\|w_1\|_{\lambda,B}\,\|1/w_2\|_{\frac{\lambda r'}{r},B} \\
&= \left(\left(\frac{1}{|B|}\int_B w_1^\lambda \mathrm{d}x\right)^{\frac{1}{\lambda r}}\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\frac{\lambda r'}{r}}\mathrm{d}x\right)^{\frac{1}{\lambda r'}}\right)^r. \qquad (3.3)
\end{aligned}
$$

Note that

$$
\sup_{B\subset E}\left(\frac{1}{|B|}\int_B w_1^\lambda \mathrm{d}x\right)^{\frac{1}{\lambda r}}\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\frac{\lambda r'}{r}}\mathrm{d}x\right)^{\frac{1}{\lambda r'}} < \infty,
$$

which yields

$$
\sup_{B\subset E}\left(\frac{1}{|B|}\int_B w_1 \mathrm{d}x\right)\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\frac{1}{r-1}}\mathrm{d}x\right)^{r-1} \le \sup_{B\subset E}\left(\left(\frac{1}{|B|}\int_B w_1^\lambda \mathrm{d}x\right)^{\frac{1}{\lambda r}}\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\frac{\lambda r'}{r}}\mathrm{d}x\right)^{\frac{1}{\lambda r'}}\right)^r < \infty.
$$

This ends the proof of Proposition 3.3. □

Theorem 3.4. *Let $\omega \in D'(M, \wedge^k)$ be a solution of the nonhomogeneous A-harmonic equation in a domain $M \subset \mathbb{R}^n$ and $\mathrm{d}\omega \in L^p(M, \wedge^{k+1})$, $k = 0, 1, \ldots, n-1$, G be Green's operator. Assume that $1 < p < \infty$ and $(w_1(x), w_2(x)) \in A_r(M)$ for some $1 < r < \infty$. Then, there is a constant C, independent of ω, such that*

$$
\|G(\omega)-(G(\omega))_B\|_{L^p(\log L)^\alpha(B,w_1^\beta)} \le C|B|diam(B)\|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(\sigma B,w_2^\beta)} \qquad (3.4)
$$

for all balls B with $\sigma B \subset M$ and $diam(B) \ge d_0$. Here $\alpha > 0$ is any constant and $\sigma > 1$, $0 < \beta < 1$ and $d_0 > 0$ are some constants.

Proof. Choose $m = p/\beta$. Then, $m > p > 1$. Using the Hölder inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{m}$, where $q = p/(1-\beta)$, we have

$$
\begin{aligned}
\|G(\omega)-(G(\omega))_B\|_{L^p(\log L)^\alpha(B,w_1^\beta)} &= \left(\int_B |G(\omega)-(G(\omega))_B|^p \log^\alpha\left(\mathrm{e}+\frac{|G(\omega)-(G(\omega))_B|}{\|G(\omega)-(G(\omega))_B\|_{p,B}}\right)w_1^\beta \mathrm{d}x\right)^{\frac{1}{p}} \\
&\le \left(\int_B |G(\omega)-(G(\omega))_B|^q \log^{\frac{\alpha q}{p}}\left(\mathrm{e}+\frac{|G(\omega)-(G(\omega))_B|}{\|G(\omega)-(G(\omega))_B\|_{p,B}}\right)\mathrm{d}x\right)^{\frac{1}{q}}\|w_1\|_{1,B}^{1/m} \\
&\le C_1\left(\int_B |G(\omega)-(G(\omega))_B|^q \log^{\frac{\alpha q}{p}}\left(\mathrm{e}+\frac{|G(\omega)-(G(\omega))_B|}{\|G(\omega)-(G(\omega))_B\|_{q,B}}\right)\mathrm{d}x\right)^{\frac{1}{q}}\|w_1\|_{1,B}^{1/m} \\
&= C_1\|G(\omega)-(G(\omega))_B\|_{L^q(\log L)^{\frac{\alpha q}{p}}(B)}\|w_1\|_{1,B}^{1/m}. \qquad (3.5)
\end{aligned}
$$

Applying Theorem 2.4 yields

$$
\|G(\omega)-(G(\omega))_B\|_{L^q(\log L)^{\frac{\alpha q}{p}}(B)} \le C_2|B|\mathrm{diam}(B)\|\mathrm{d}\omega\|_{L^q(\log L)^{\frac{\alpha q}{p}}(\sigma_1 B)},
$$

where $\sigma_1 > 1$ is some constant. Let $s = \frac{p}{\beta(r-1)}$ and $t = \frac{p}{1+\beta(r-1)}$. Using Lemma 2.2, we have

$$
\|\mathrm{d}\omega\|_{L^q(\log L)^{\frac{\alpha q}{p}}(\sigma_1 B)} \le C_3|B|^{\frac{t-q}{qt}}\|\mathrm{d}\omega\|_{L^t(\log L)^{\frac{\alpha t}{p}}(\sigma_2 B)} \qquad (3.6)
$$

for some $\sigma_2 > \sigma_1$. Note that it is easy to check that $\frac{1}{t} = \frac{1}{p} + \frac{1}{s}$. Thus, using the Hölder inequality, we have

$$
\begin{aligned}
\|\mathrm{d}\omega\|_{L^t(\log L)^{\frac{\alpha t}{p}}(\sigma_2 B)} &= \left(\int_{\sigma_2 B}|\mathrm{d}\omega|^t \log^{\frac{\alpha t}{p}}\left(\mathrm{e}+\frac{|\mathrm{d}\omega|}{\|\mathrm{d}\omega\|_{t,\sigma_2 B}}\right)w_2^{\frac{t\beta}{p}}\left(\frac{1}{w_2}\right)^{\frac{t\beta}{p}}\mathrm{d}x\right)^{\frac{1}{t}} \\
&\le \left(\int_{\sigma_2 B}|\mathrm{d}\omega|^p \log^{\alpha}\left(\mathrm{e}+\frac{|\mathrm{d}\omega|}{\|\mathrm{d}\omega\|_{t,\sigma_2 B}}\right)w_2^{\beta}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\sigma_2 B}\left(\frac{1}{w_2}\right)^{\frac{s\beta}{p}}\mathrm{d}x\right)^{\frac{1}{s}}
\end{aligned}
$$

$$\leq C_4 \left(\int_{\sigma_2 B} |\mathrm{d}\omega|^p \log^\alpha \left(\mathrm{e} + \frac{|\mathrm{d}\omega|}{\|\mathrm{d}\omega\|_{p,\sigma_2 B}} \right) w_2^\beta \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\sigma_2 B} \left(\frac{1}{w_2} \right)^{\frac{s\beta}{p}} \mathrm{d}x \right)^{\frac{1}{s}}$$

$$= C_4 \|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(\sigma_2 B, w_2^\beta)} \left(\int_{\sigma_2 B} \left(\frac{1}{w_2} \right)^{\frac{s\beta}{p}} \mathrm{d}x \right)^{\frac{1}{s}}. \tag{3.7}$$

Note that $(w_1(x), w_2(x)) \in A_r(M)$, then

$$\begin{aligned}
\left(\int_{\sigma_2 B} \left(\frac{1}{w_2} \right)^{\frac{s\beta}{p}} \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_B w_1 \mathrm{d}x \right)^{\frac{1}{m}} &\leq \left(\int_{\sigma_2 B} \left(\frac{1}{w_2} \right)^{\frac{s\beta}{p}} \mathrm{d}x \right)^{\frac{1}{s}} \left(\int_{\sigma_2 B} w_1 \mathrm{d}x \right)^{\frac{1}{m}} \\
&= |\sigma_2 B|^{\frac{1}{s}+\frac{1}{m}} \left(\frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} \left(\frac{1}{w_2} \right)^{\frac{1}{r-1}} \mathrm{d}x \right)^{\frac{\beta(r-1)}{p}} \left(\frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} w_1 \mathrm{d}x \right)^{\frac{\beta}{p}} \\
&= |\sigma_2 B|^{\frac{1}{s}+\frac{1}{m}} \left(\left(\frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} \left(\frac{1}{w_2} \right)^{\frac{1}{r-1}} \mathrm{d}x \right)^{r-1} \left(\frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} w_1 \mathrm{d}x \right) \right)^{\frac{\beta}{p}} \\
&\leq C_5 |B|^{\frac{1}{s}+\frac{1}{m}}.
\end{aligned} \tag{3.8}$$

Combining (3.6)–(3.8), we have

$$\begin{aligned}
\|G(\omega) - (G(\omega))_B\|_{L^p(\log L)^\alpha(B, w_1^\beta)} &\leq C_6 |B| \mathrm{diam}(B) |B|^{\frac{t-q}{qt}} |B|^{\frac{1}{s}+\frac{1}{m}} \|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(\sigma_2 B, w_2^\beta)} \\
&= C_6 |B| \mathrm{diam}(B) \|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(\sigma_2 B, w_2^\beta)}.
\end{aligned} \tag{3.9}$$

This ends the proof of Theorem 3.4. □

The following definition of $L^\varphi(\mu)$-domains was introduced in [12].

Definition 3.5. Let φ be a Young function on $[0, \infty)$ with $\varphi(0) = 0$. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L^\varphi(\mu)$-domain, if there exists a constant C such that

$$\int_\Omega \varphi(\sigma|\omega - \omega_\Omega|)\mathrm{d}\mu \leq C \sup_{B \subset \Omega} \int_B \varphi(\tau|\omega - \omega_B|)\mathrm{d}\mu \tag{3.10}$$

for all ω such that $\varphi(|\omega|) \in L^1_{loc}(\Omega, \mu)$, where the measure μ is defined by $\mathrm{d}\mu = w(x)\mathrm{d}x$, $w(x)$ is a weight and τ, σ are constants with $0 < \tau \leq 1$, $0 < \sigma \leq 1$ and the supremum is over all balls $B \subset \Omega$.

Theorem 3.6. *Assume G is Green's operator and $\Omega \subset \mathbb{R}^n$ is a bounded $L^\varphi(\mu)$-domain with $\varphi(t) = t^p \log^\alpha(\mathrm{e} + \frac{t}{c})$, where $c = \|G(\omega) - (G(\omega))_{B_0}\|_{p,\Omega}$, $1 < p < \infty$, and $B_0 \subset \Omega$ is a fixed ball. Let $\omega \in D'(\Omega, \wedge^0)$ be a solution of the nonhomogeneous A-harmonic equation in Ω, $\mathrm{d}\omega \in L^p(\Omega, \wedge^1)$ and $(w_1(x), w_2(x)) \in A_r(\Omega)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of ω, such that*

$$\|G(\omega) - (G(\omega))_\Omega\|_{L^p(\log L)^\alpha(\Omega, w_1^\beta)} \leq C|\Omega| diam(\Omega) \|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(\Omega, w_2^\beta)}, \tag{3.11}$$

where β is a constant with $0 < \beta \leq 1$.

Proof. It is clear that for any constants $k_i > 0$, $i = 1, 2, 3$, there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \log\left(\mathrm{e} + \frac{t}{k_1} \right) \leq \log\left(\mathrm{e} + \frac{t}{k_2} \right) \leq C_2 \log\left(\mathrm{e} + \frac{t}{k_3} \right) \tag{3.12}$$

for any $t > 0$. Therefore, using the definition of $L^\varphi(\mu)$-domains with $\tau = 1, \sigma = 1$ and $\varphi(t) = t^p \log^\alpha(\mathrm{e} + \frac{t}{c})$, where $c = \|G(\omega) - (G(\omega))_{B_0}\|_{p,\Omega}$, Theorem 3.4 and (3.12), we obtain

$$\begin{aligned}
\|G(\omega) - (G(\omega))_\Omega\|^p_{L^p(\log L)^\alpha(\Omega, w_1^\beta)} &= \int_\Omega |G(\omega) - (G(\omega))_\Omega|^p \log^\alpha \left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_\Omega|}{\|G(\omega) - (G(\omega))_\Omega\|_{p,\Omega}} \right) w_1^\beta \mathrm{d}x \\
&\leq C_1 \int_\Omega |G(\omega) - (G(\omega))_\Omega|^p \log^\alpha \left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_\Omega|}{\|G(\omega) - (G(\omega))_{B_0}\|_{p,\Omega}} \right) w_1^\beta \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le C_2 \sup_{B\subset\Omega} \int_B |G(\omega) - (G(\omega))_B|^p \log^\alpha \left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_{B_0}\|_{p,\Omega}} \right) w_1^\beta \mathrm{d}x \\
&\le C_3 \sup_{B\subset\Omega} \int_B |G(\omega) - (G(\omega))_B|^p \log^\alpha \left(\mathrm{e} + \frac{|G(\omega) - (G(\omega))_B|}{\|G(\omega) - (G(\omega))_B\|_{p,B}} \right) w_1^\beta \mathrm{d}x \\
&\le C_4 \sup_{B\subset\Omega} |B|^p \mathrm{diam}^p(B) \|\mathrm{d}\omega\|^p_{L^p(\log L)^\alpha(\sigma B, w_2^\beta)} \\
&\le C_4 \sup_{B\subset\Omega} |\Omega|^p \mathrm{diam}^p(\Omega) \|\mathrm{d}\omega\|^p_{L^p(\log L)^\alpha(\Omega, w_2^\beta)} \\
&= C_4 |\Omega|^p \mathrm{diam}^p(\Omega) \|\mathrm{d}\omega\|^p_{L^p(\log L)^\alpha(\Omega, w_2^\beta)},
\end{aligned}
\tag{3.13}
$$

which yields

$$
\|G(\omega) - (G(\omega))_\Omega\|_{L^p(\log L)^\alpha(\Omega, w_1^\beta)} \le C|\Omega| \mathrm{diam}(\Omega) \|\mathrm{d}\omega\|_{L^p(\log L)^\alpha(\Omega, w_2^\beta)}.
$$

We have completed the proof of Theorem 3.6. □

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References

[1] M.P. do Carmo, Differential Forms and Applications, Springer-Verlag, Berlin, 1994.
[2] R.P. Agarwal, S. Ding, C.A. Nolder, Inequalities for Differential Forms, Springer, New York, 2009.
[3] Y. Wang, C. Wu, Sobolev imbedding theorems and Poincaré inequalities for Green's operator on solutions of the nonhomogeneous A-harmonic equation, Comput. Math. Appl. 47 (2004) 1545–1554.
[4] Y. Xing, C. Wu, Global weighted inequalities for operators and harmonic forms on manifolds, J. Math. Anal. Appl. 294 (2004) 294–309.
[5] S. Ding, B. Liu, Global integrability of the Jacobians of a composite mapping, J. Inequal. Appl. 2006 (2006).
[6] S. Ding, Norm estimates for the maximal operator and Green's operator, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 16 (2009). Differential Equations Dynam. Systems, suppl. S1, 72–78.
[7] C.A. Nolder, Global integrability theorems for A-harmonic tensors, J. Math. Anal. Appl. 247 (2000) 236–245.
[8] B. Liu, The monotonic property of $L^s(\mu)$-averaging domains and weighted weak reverse Hölder inequality, J. Math. Anal. Appl. 237 (1999) 730–739.
[9] R.P. Agarwal, S. Ding, Global Caccioppoli-type and Poincaré inequalities with Orlicz norms, J. Inequal. Appl. 2010 (2010) 27, Article ID 727954.
[10] C.J. Neugebauer, Inserting A_p-weights, Proc. Amer. Math. Soc. 87 (1993) 644–648.
[11] J. Heinonen, T. Kilpelainen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford University Press, Oxford, 1993.
[12] S. Ding, $L^\varphi(\mu)$-averaging domains and the quasihyperbolic metric, Comput. Math. Appl. 47 (2004) 1611–1618.